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The oscillatory instability of convection rolls in a low Prandtl number fluid

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The instability of convection rolls in a fluid layer heated from below is investigated for stress-free boundaries in the limit of small Prandtl number. It is shown that the two-dimensional rolls become unstable to oscillatory three-dimensional disturbances when the amplitude of the convective motion exceeds a finite critical value. The instability corresponds to the generation of vertical vorticity, a mechanism which is likely to operate in the case of a variety of roll-like motions. In all aspects in which the theory can be related to experiments, reasonable agreement with the observations is found.

1. Introduction

The onset of convection in a layer heated from below is a particularly simple example of hydrodynamic instability. Because the principle of exchange of stabilities is valid, i.e. the instability of the static layer occurs in the form of non-oscillatory small amplitude motions, a detailed investigation of the non-linear post-instability problem has been possible. Owing to the progress in the understanding of its finite amplitude properties, convection has become a key problem in the theory of the transition from laminar to turbulent flow. In contrast to the sudden occurrence of turbulence in the case of plane parallel shear flow, the transition in the case of convection is characterized by a series of subsequent instabilities as the amplitude is increased. In each of the instabilities a more complex form of convection replaces the previous type of convection. This process has been studied most extensively in the case of high Prandtl number and reasonable agreement between experimental observations and theoretical predictions has been found (Busse & Whitehead 1971). In the present paper we shall be concerned with the transition of convection rolls to a more complex form of flow in the complementary case of small Prandtl numbers. Although the analysis is restricted by the assumption of stress-free boundaries, it can be expected that the results found in the two limiting extremes of the Prandtl number will yield a qualitatively complete picture of the instabilities of convection rolls.

There also have been other motivations for this paper. One of the most puzzling features of convection is the occurrence of oscillatory convection. In experiments with a layer of mercury it has been observed by Rossby (1969) and others that the convective motions always exhibit a non-stationary oscillatory time dependence in contrast to the steady convection observed in non-metallic fluids at sufficiently small Rayleigh numbers. This difference must be attributed to the low Prandtl number of mercury. The theoretical analysis, on the other hand, has not given any indication hitherto of a qualitative dependence of convection on the Prandtl number. In the non-dimensional description of the problem based on the Boussinesq equations the Prandtl number appears as the only physical parameter apart from the Rayleigh number. The onset of convection is independent of the Prandtl number, however, and even for Rayleigh numbers beyond the critical value only a quantitative dependence on the Prandtl number has been found. It will be shown in this paper that the usual analysis based on an expansion in powers of the amplitude has limited relevance in the limit of low Prandtl number since it neglects some important nonlinear effects. For this reason a different analytical approach will be adopted in the following in order to isolate the mechanism of oscillatory instability.

To illustrate the last remark, it is of interest to consider the peculiar property of the convective instability that all unstable modes lack a vertical component of vorticity. In fact, it has been shown by Schlüter, Lortz & Busse (1965) (whose paper we shall refer to as SLB) that the vertical vorticity of all possible solutions of the equations of motion vanishes like A^m with $m \geqslant 3$, as the amplitude A of convection tends to zero. Since it has been proved in SLB that convection in the form of two-dimensional rolls represents the only stable form of convection at sufficiently small amplitudes and since the vertical vorticity vanishes for this solution for reasons of symmetry, it appeared that the vertical vorticity did not play a role in convection processes. It would be surprising, however, if the dynamics of convection should lack at all amplitudes the additional degree of freedom which the vertical component of vorticity provides. The nonlinear terms in the equation of motion which are responsible for the generation of vertical vorticity increase in importance relative to the nonlinear terms in the heat equation as the Prandtl number decreases. Hence we expect that the instability of convection rolls is most likely to be associated with vertical vorticity in the case of low Prandtl number. The following analysis confirms this expectation and shows that the instability is characterized by the interaction of vertical vorticity and oscillatory time dependence. Thus the analysis combines the two elements which have been missing so far in the theory of convection.

The formulation of the stability problem and the method of solution are outlined in § 2. For simplicity attention will be restricted to the case of stress-free boundaries with fixed temperatures at top and bottom of the fluid layer. This case allows an analytical treatment of the problem with minimal reliance on numerical computations. We shall consider disturbances which depend weakly on the y co-ordinate in the direction of the axis of the convection rolls. The weak dependence permits the expansion of the dependent variables in powers of the wavenumber b in the y direction. The hierarchy of linear inhomogeneous equations induced by the expansion will be analysed in § 3. The equations of order b require the disturbances to have an oscillating time dependence with a frequency proportional to the amplitude of the stationary convection roll. To determine the real part of the growth rate the equations of order b^2 have to be considered. The result shows that growing disturbances can exist when the amplitude A exceeds a critical value A_i. In order to extend the stability analysis to disturbances with larger wavenumbers we use a different approach in § 4, based on a Galerkin procedure. The numerical calculations show the anticipated result that the amplitude A_i of marginal stability is determined by disturbances with nearly vanishing wavenumber b. Disturbances of maximum growth, however, correspond to finite values of b as soon as A exceeds the critical value A_i. A discussion of the results and a comparison with related experimental observations follows in § 5.

2. Formulation of the problem

The mathematical description of convection will be based on the Boussinesq approximation of the equations of motion and the heat equation. Accordingly, all material properties are assumed to be constant with the exception of the density in the gravity term. Using the thickness d, d^2/ν and the temperature difference between lower and upper boundary divided by RP^{-1} as units for length, time and temperature, respectively, we obtain the following dimensionless equations for the velocity vector $\mathbf{v}$ and the deviation θ of the temperature from the distribution in the static state:

$$\nabla^2\mathbf{v} + \boldsymbol{\lambda}\theta - \nabla p = \mathbf{v}\,.\,\nabla\mathbf{v} + \partial\mathbf{v}/\partial t, \tag{2.1}$$

$$\nabla\,.\,\mathbf{v} = 0, \tag{2.2}$$

$$\nabla^2\theta + R\boldsymbol{\lambda}\,.\,\mathbf{v} = P(\mathbf{v}\,.\,\nabla\theta + \partial\theta/\partial t). \tag{2.3}$$

R is the Rayleigh number and $P \equiv \nu/\kappa$ is the Prandtl number. Since the time scale is based on the viscous diffusion time d^2/ν rather than on the thermal diffusion time d^2/κ, as in the usual formulation of the problem (e.g. Busse 1967), the Prandtl number appears in the heat equation and not in the equation of motion. $\boldsymbol{\lambda}$ denotes the unit vector opposite to the direction of gravity.

In order to eliminate the equation of continuity we introduce the following general representation of the divergence-free velocity field:

$$\mathbf{v} = \boldsymbol{\delta}v + \boldsymbol{\epsilon}w, \tag{2.4}$$

where the operators $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$ are defined by

$$\boldsymbol{\delta}v \equiv \nabla\times(\nabla\times\boldsymbol{\lambda}v), \quad \boldsymbol{\epsilon}w \equiv \nabla\times\boldsymbol{\lambda}w.$$

For the scalar functions v, w, and θ we obtain the following equations from (2.1) and (2.3):

$$\left.\begin{aligned} \nabla^4\Delta_2 v - \Delta_2\theta &= \boldsymbol{\delta}\,.\{(\boldsymbol{\delta}v + \boldsymbol{\epsilon}w)\,.\,\nabla(\boldsymbol{\delta}v + \boldsymbol{\epsilon}w)\} + \frac{\partial}{\partial t}(\nabla^2\Delta_2 v), \\ \nabla^2\Delta_2 w &= \boldsymbol{\epsilon}\,.\{(\boldsymbol{\delta}v + \boldsymbol{\epsilon}w)\,.\,\nabla(\boldsymbol{\delta}v + \boldsymbol{\epsilon}w)\} + \frac{\partial}{\partial t}(\Delta_2 w), \\ \nabla^2\theta - R\Delta_2 v &= P\left\{(\boldsymbol{\delta}v + \boldsymbol{\epsilon}w)\,.\,\nabla\theta + \frac{\partial}{\partial t}\theta\right\}, \end{aligned}\right\} \tag{2.5}$$

where Δ_2 is the Laplacian with respect to horizontal dimensions. We shall use a Cartesian system of co-ordinates with the z co-ordinate in the direction of $\boldsymbol{\lambda}$ and the origin at the lower boundary. Since the boundaries have fixed temperatures and do not exert stresses in the horizontal directions, the boundary conditions for v, w and θ are

$$v = \partial^2_{zz} v = \partial_z w = \theta = 0 \quad \text{at} \quad z = 0, 1, \tag{2.6}$$

where the symbol ∂_z denotes the partial derivative with respect to z, ∂^2_{zz} denoting the second derivative.

The stationary two-dimensional solution of equations (2.5) corresponding to convection in the form of rolls can be represented by the following series in powers of the Prandtl number P:

$$\left.\begin{aligned} v &= A\{\cos\alpha x \sin\pi z + o(A^2P^2)\},\\ \theta &= A\{(\pi^2+\alpha^2)^2\cos\alpha x\sin\pi z - PA(\pi^2+\alpha^2)^2\alpha^2(4\pi)^{-1}\sin 2\pi z + o(A^2P^2)\},\\ w &= 0,\\ R &= (\pi^2+\alpha^2)^3/\alpha^2 + \tfrac{1}{8}P^2\alpha^2A^2(\pi^2+\alpha^2)^2\ldots. \end{aligned}\right\} \tag{2.7}$$

For details of the derivation we refer to SLB. The wavenumber α is an independent parameter of the solution (2.7). Of physical importance is the value $\alpha = \alpha_c \equiv \pi/\sqrt{2}$, for which the Rayleigh number reaches its lowest value R_c as a function of α and A. R_c is the critical value at which the static solution of the problem $v = \theta = w = 0$ becomes unstable when the temperature difference between the boundaries is slowly increased. Although only the case $\alpha = \alpha_c$ has physical relevance we shall leave α unspecified in most of the following analysis. Since the terms which have not been denoted explicitly in (2.7) are proportional to P^2 or to higher powers of the Prandtl number the expressions (2.7) describe an exact solution of the nonlinear equations (2.5) in the limit

$$(PA)^2 \ll 1. \tag{2.8}$$

The following analysis will take advantage of this fact by assuming the case of vanishing P^2, while the amplitude A remains finite. Terms proportional to P will be taken into account to indicate the influence of the Prandtl number. Although the term of the order P^2A^2 in the expression for R in (2.7) will not be taken into account in the analysis, it has been denoted explicitly to obtain an approximate relation between A and R for the comparison with experimental results.

In order to investigate the stability of the solution (2.7) we superimpose infinitesimal disturbances of arbitrary spatial dependence. Since the equations of the disturbances $\hat{v}$, $\hat{w}$, $\hat{\theta}$ are linear and do not depend explicitly on t and y an exponential dependence on these variables can be assumed:

$$\left.\begin{aligned} \hat{v} &= \tilde{v}(x,z)\exp\{iby+\sigma t\},\\ \hat{w} &= \tilde{w}(x,z)\exp\{iby+\sigma t\},\\ \hat{\theta} &= \tilde{\theta}(x,z)\exp\{iby+\sigma t\}. \end{aligned}\right\} \tag{2.9}$$

Because the disturbances have to be bounded at infinity, the wavenumber b must be real. Inserting expressions (2.9) in (2.7) we obtain as equations for $\tilde{v}$, $\tilde{w}$, $\tilde{\theta}$

$$
\begin{aligned}
(\nabla^4-\sigma\nabla^2)\,\Delta_2\tilde{v}-\Delta_2\tilde{\theta} = {} & \boldsymbol{\delta}.\{\boldsymbol{\delta}\tilde{v}.\nabla\boldsymbol{\delta}v+\boldsymbol{\delta}v.\nabla\boldsymbol{\delta}\tilde{v}\}+ib\{\partial^2_{xz}(\boldsymbol{\delta}v.\nabla\tilde{w})+\boldsymbol{\delta}.(\tilde{w}\,\partial_x\boldsymbol{\delta}v) \\
& -\partial_z(\boldsymbol{\delta}v.\nabla\partial_x\tilde{w})\}-b^2\{\partial_z(\boldsymbol{\delta}v.\nabla\partial_z\tilde{v}) \\
& +\boldsymbol{\delta}v.\nabla\Delta_2\tilde{v}+\Delta_2(\boldsymbol{\delta}v.\nabla\tilde{v})-\boldsymbol{\delta}.(\tilde{v}\,\partial_z\boldsymbol{\delta}v) \\
& +\boldsymbol{\delta}\tilde{v}.\nabla\Delta_2 v-(\nabla^4+2\nabla^2\Delta_2-\sigma\nabla^2-\sigma\Delta_2)\,\tilde{v}+\tilde{\theta}\} \\
& -ib^3\tilde{w}\,\partial_x\Delta_2 v+b^4\{\boldsymbol{\delta}v.\nabla\tilde{v}-\tilde{v}\,\partial_z\Delta_2 v \\
& -(\Delta_2+2\nabla^2-\sigma)\,\tilde{v}\}+b^6\tilde{v}, \qquad (2.10a)
\end{aligned}
$$

$$
\begin{aligned}
(\nabla^2-\sigma)\,\Delta_2\,\tilde{w} = {} & \partial_x(\boldsymbol{\delta}v.\nabla\partial_x\,\tilde{w})+ib\{\partial_x\Delta_2 v\,\partial^2_{zz}\tilde{v}-\partial^2_{zz}\partial_x v\Delta_2\tilde{v}\} \\
& -b^2\{\partial_x(\tilde{w}\,\partial^2_{xz}v)-\Delta_2 v\,\partial_z\,\tilde{w}-(\nabla^2+\Delta_2-\sigma)\,\tilde{w}\} \\
& +ib^3\tilde{v}\,\partial^2_{zz}\partial_x v-b^4\tilde{w}, \qquad (2.10b)
\end{aligned}
$$

$$
\nabla^2\tilde{\theta}-R\Delta_2\tilde{v} = P\{\sigma\tilde{\theta}+\boldsymbol{\delta}\tilde{v}.\nabla\theta+\boldsymbol{\delta}v.\nabla\tilde{\theta}+ib\tilde{w}\,\partial_x\theta+b^2\tilde{v}\,\partial_z\theta\}+b^2(\tilde{\theta}-R\tilde{v}). \qquad (2.10c)
$$

The stability analysis in SLB has shown that the two-dimensional solution (2.7) with $\alpha=\alpha_c$ is the only stable solution of equations (2.5) for sufficiently small values of A. All possible disturbances correspond to real values of the growth rate σ in this case. The disturbance with highest growth rate is given by

$$
\tilde{v}=v_0\equiv(-1/\alpha A)\,\partial_x v,\quad \tilde{\theta}=\theta_0\equiv(-1/\alpha A)\,\partial_x\theta,\quad \sigma=0,\quad \partial_x\,\tilde{w}=0, \qquad (2.11)
$$

while all other disturbances have negative values of σ. The present analysis is concerned with the stability of solution (2.7) for values A of the order unity and larger, in the range described by the limit (2.8). We expect that the most destabilizing disturbance will be closely related to the particular disturbance (2.11). By differentiating equations (2.5) with respect to x it can readily be verified that (2.11) represents an exact solution of (2.10) with $b=0$ for arbitrary values of the amplitude A. The physical meaning of the disturbance (2.11) is the translation of the stationary convection pattern in the x direction. The vanishing growth rate σ reflects the translational invariance of convection in a horizontally infinite layer.

To investigate the possibility of disturbances with a positive real part of σ in the neighbourhood of the solution (2.11) we shall consider disturbances of the form (2.9) for small values of b. This allows us to assume an expansion in powers of b:

$$
\left.
\begin{aligned}
\tilde{v} &= v_0+bv_1+b^2v_2+\ldots, \\
\tilde{\theta} &= \theta_0+b\theta_1+b^2\theta_2+\ldots, \\
\sigma &= \sigma_0+b\sigma_1+b^2\sigma_2+\ldots.
\end{aligned}
\right\} \qquad (2.12)
$$

Before we write down the expansion for the variable $\tilde{w}$ we have to pay attention to the different ways in which the x-dependent part w' and the x-independent part $\overline{w}=\tilde{w}-w'$ of the variable $\tilde{w}$ enter the problem. For the latter part, the terms $\{\partial_x\Delta_2 v\,\partial^2_{zz}\tilde{v}-\partial^2_{zz}\partial_x v\,\Delta_2\tilde{v}\}$ in (2.10b) vanish, which suggests that the expansions for $\overline{w}$ and for w' should be based on different equations. We also note that the x-independent part of $\tilde{w}$ remains undetermined in the solution (2.11) since only the x derivative of $\tilde{w}$ has physical meaning in the two-dimensional case $b = 0$. In principle we have to admit the possibility

$$\overline{w} \approx O(b^{-1}) \quad \text{for} \quad b \to 0 \tag{2.13}$$

since such a function $\overline{w}$ would still lead to a finite expression for the velocity of the disturbance. Hence we shall assume

$$\left.\begin{aligned} \overline{w} &= b^{-1}\{\overline{w}_0 + b\overline{w}_1 + b^2 w_2 + \dots\}, \\ w' &= w_0' + bw_1' + b^2 w_2' + \dots . \end{aligned}\right\} \tag{2.14}$$

The analysis of the next section will show that $\overline{w}_0$, w_0' and σ_0 have to vanish, thus establishing (2.11) as an analytical solution of (2.10) in the limit $b \to 0$.

3. The oscillatory instability

By inserting expressions (2.12) and (2.14) into the system of equations (2.10) we obtain a hierarchy of linear equations corresponding to different powers of b. The equations are inhomogeneous with the exception of the equations of lowest order in b which have the form

$$(\nabla^2 - \sigma_0)\nabla^2\Delta_2 v_0 - \Delta_2\theta_0 - V_a v_0 - iV_b \overline{w}_0 = 0, \tag{3.1a}$$

$$(\partial^2_{zz} - \sigma_0)\overline{w}_0 + iV_c v_0 = 0, \tag{3.1b}$$

$$(\nabla^2 - \sigma_0)\Delta_2 w_0' - \partial_x(\boldsymbol{\delta} v . \nabla\partial_x w_0') = 0, \tag{3.1c}$$

$$\nabla^2\theta_0 - R\Delta_2 v_0 - P\{\sigma\theta_0 + \boldsymbol{\delta} v_0 . \nabla\theta + \boldsymbol{\delta} v . \nabla\theta_0 + i\overline{w}_0\partial_x\theta\} = 0. \tag{3.1d}$$

The following operators have been introduced for convenience:

$$\left.\begin{aligned} V_a v_0 &\equiv \boldsymbol{\delta} . \{\boldsymbol{\delta} v . \nabla\boldsymbol{\delta} v_0 + \boldsymbol{\delta} v_0 . \nabla\boldsymbol{\delta} v\}, \\ V_b \overline{w}_0 &\equiv \overline{w}_0 \partial_x \nabla^2\Delta_2 v - \partial^2_{zz}\overline{w}_0 \partial_x \Delta_2 v, \\ V_c v_0 &\equiv \partial_z\overline{\{\partial_z v_0 \partial_x \Delta_2 v - v_0 \partial^2_{zx}\Delta_2 v\}}. \end{aligned}\right\} \tag{3.2}$$

The average over the x co-ordinate is indicated by a bar. The total average over the z as well as the x co-ordinate will be indicated by angular brackets $\langle \dots \rangle$.

By multiplying (3.1c) by w_0' and averaging it we obtain

$$\langle w_0'(\nabla^2 - \sigma_0)\Delta_2 w_0'\rangle + \tfrac{1}{2}\langle\nabla . (\boldsymbol{\delta} v(\partial_x w_0')^2)\rangle = \langle(\nabla\partial_x w_0')^2 + \sigma_0(\partial_x w_0')^2\rangle = 0.$$

Disregarding the possibility of negative values of σ_0, we conclude that $w_0' = 0$. Another condition on the solution of (3.1) is found by taking the z average of (3.1b), which yields $\sigma_0\langle w_0\rangle = 0$. Inspection of (3.1) shows that solution (2.11) solves the equation with vanishing $\overline{w}_0$. In addition to the solution

$$v_0 = (-1/\alpha A)\partial_x v, \quad \overline{w}_0 = w_0' = \sigma_0 = 0, \quad \theta_0 = (-1/\alpha A)\partial_x\theta, \tag{3.3}$$

the solution

$$v_0 = (1/A)v, \quad \overline{w}_0 = w_0' = \sigma_0 = 0, \quad \theta_0 = (1/A)\theta \tag{3.4}$$

is also possible. The latter solution does not lead to instability, however, when terms of higher order in b are considered, and has negative values of σ_0 when terms of the order A^2P^2 are taken into account in contrast to solution (3.3), which always corresponds to $\sigma_0 = 0$. From the analysis in SLB it is known that (3.3) and (3.4) are the only solutions of (3.1) with a non-negative real part of σ_0 for sufficiently small values of A. We expect that this fact holds for arbitrary values of A, although we shall not attempt to prove this here since we are not interested in the general problem of stability. For the purpose of proving instability it will be sufficient to restrict the attention to disturbances which can be regarded as small modifications of the solution (3.3). By starting with expressions (3.3) the inhomogeneous equations of higher order in b can be solved subsequently. In order to determine the solutions of the inhomogeneous equations uniquely we impose as an orthogonality condition

$$R\langle v_0 \nabla^2 \Delta_2 v_n\rangle + P\langle \theta_0 \theta_n\rangle = 0 \quad \text{for} \quad n \geqslant 1. \tag{3.5}$$

Since σ_0 vanishes, the average $\langle w_n\rangle$ of $\tilde{w}_n$ will be determined by equations of higher order in b than the equation for $\overline{w}_n$. For this reason $\overline{w}_n$ will be separated into the two parts

$$\overline{w}_n = \langle w_n\rangle + \ddot{w}_n' \quad \text{with} \quad \langle w_n'\rangle = 0 \quad \text{for} \quad n \geqslant 1.$$

The equations of first order in b have the form

$$\nabla^4 \Delta_2 \theta_1 - \Delta_2 \theta_1 - V_a v_1 - iV_b \overline{w}_1' = \sigma_1 \nabla^2 \Delta_2 v_0 + i\langle w_1\rangle \partial_x \nabla^2 \Delta_2 v, \tag{3.6a}$$

$$\partial_{zz}^2 \overline{w}_1' + iV_c v_1 = 0, \tag{3.6b}$$

$$\nabla^2 \Delta_2 w_1' - \partial_x(\boldsymbol{\delta} v \cdot \nabla \partial_x w_1') = 0, \tag{3.6c}$$

$$\nabla^2 \theta_0 - R\Delta_2 v_0 - P\{\boldsymbol{\delta} v_1 . \nabla \theta + \boldsymbol{\delta} v . \nabla \theta_1\} = P(\sigma_1 \theta_0 + i\langle w_1\rangle \partial_x \theta). \tag{3.6d}$$

This system of equations reduces to the homogeneous system of equations when

$$\sigma_1 = i\langle w_1\rangle A\alpha \tag{3.7}$$

is assumed. Hence we conclude in accordance with condition (3.5) that

$$v_1 = w_1' = \ddot{w}_1' = \theta_1 = 0. \tag{3.8}$$

In order to determine $\langle w_1\rangle$ we have to consider the total average of (2.10*b*)

$$(\sigma + b^2)\, b^2 \langle \tilde{w}\rangle = b^2 \langle \Delta_2 v\, \partial_z \tilde{w}\rangle + ib^3 \langle \tilde{v}\, \partial_{zz}^2 \partial_x v\rangle. \tag{3.9}$$

Comparison of the coefficients of b^n in this relation yields the first non-trivial result for $n = 3$:

$$\sigma_1 \langle w_1\rangle = i\langle v_0 \partial_{zz}^2 \partial_x v\rangle = i\tfrac{1}{4}\alpha\pi^2 A. \tag{3.10}$$

This relation together with (3.7) gives

$$\langle w_1\rangle = \pm \tfrac{1}{2}\pi, \quad \sigma_1 = \pm i\pi A\alpha/2. \tag{3.11}$$

Since σ_1 is imaginary, higher order terms of σ must be determined to decide the problem of instability. From the terms of second order in b the following system of equations is found:

$$\begin{aligned}\nabla^4 \Delta_2 v_2 - \Delta_2 \theta_2 - V_a v_2 - iV_b \ddot{w}_2' = {} & -\partial_z(\boldsymbol{\delta} v . \nabla \partial_z v_0) + \boldsymbol{\delta} . (v_0 \partial_z \boldsymbol{\delta} v) + (\nabla^4 + 2\nabla^2 \Delta_2)\, v_0 \\ & + \sigma_2 \nabla^2 \Delta_2 v_0 + i\langle w_2\rangle \partial_x \nabla^2 \Delta_2 v - \theta_0,\end{aligned} \tag{3.12a}$$

$$\partial_{zz}^2 \overline{w}_2' + iV_c v_2 = i(\overline{v_0 \partial_{zz}^2 \partial_x v} - \langle v_0 \partial_{zz}^2 \partial_x v\rangle), \tag{3.12b}$$

$$\nabla^2 \Delta_2 w_2' - \partial_x(\boldsymbol{\delta} v . \nabla \partial_x w_2') = -\langle w_1\rangle \partial_z \Delta_2 v, \tag{3.12c}$$

$$\nabla^2 \theta_2 - R\Delta_2 v_2 - P\{\boldsymbol{\delta} v_2 . \nabla \theta + \boldsymbol{\delta} v . \nabla \theta_2\} = P\{v_0 \partial_z \theta + \sigma_2 \theta_0 + i\langle w_2\rangle \partial_x \theta\} + \theta_0 - Rv_0. \tag{3.12d}$$

The system (3.12a, b, d) of three coupled inhomogeneous linear equations can be solved when the inhomogeneity is orthogonal to the solutions of the adjoint homogeneous problem. This solvability condition provides the equation for the determination of σ_2. The relevant solution of the adjoint problem for which the solvability condition yields a non-trivial result has been derived in appendix A. Multiplication of (3.12a, b, d) by v^*, $\overline{w}^*$ and $(1/R)\,\theta^*$, respectively, adding the results and averaging yields

$$0 = \langle \partial_z v^* \boldsymbol{\delta} v \,.\, \nabla \partial_z v_0 \rangle + \langle \boldsymbol{\delta} v^* \,.\, v_0 \partial_z \boldsymbol{\delta} v \rangle + (2\alpha^2 - \pi^2)(\pi^2 + \alpha^2)\langle v^* v_0 \rangle + i \langle \overline{w}^* v_0 \partial^2_{zz} \partial_x v \rangle + (P/R)\langle \theta^* v_0 \partial_z \theta \rangle + \langle \{(\pi^2 + \alpha^2)\,\alpha^2 v^* + P\alpha^2 \theta^*/(\pi^2 + \alpha^2)\}(\sigma_2 v_0 + i\langle w_2 \rangle \partial_x v)\rangle. \tag{3.13}$$

Before this condition can be evaluated we have to go back to equation (3.9) to obtain an expression for $\langle w_2 \rangle$. From terms of order b^4 (3.9) yields the relation

$$(\sigma_2 + 1)\langle w_1 \rangle + \sigma_1 \langle w_2 \rangle = \langle \Delta_2 v \,\partial_z w_2' \rangle. \tag{3.14}$$

Since σ_1, as well as w_2', is proportional to $\langle w_1 \rangle$ this term can be cancelled from relation (3.14). By using the solution (B 1) for w_2' from appendix B, (3.14) can be rewritten in the form

$$\sigma_2 + iA\alpha\langle w_2 \rangle = -1 + \tfrac{1}{4}\alpha^2 \pi A c_{11}. \tag{3.15}$$

The solvability condition (3.13) can now be evaluated. After inserting the expressions (A 2), (A 3) and (A 5) (see appendix A) for v^*, θ^* and $\overline{w}^*$, and using (3.15) we obtain

$$2\sigma_2(1+P)(\pi^2+\alpha^2)\,\alpha^2 = [\pi^2 - \alpha^2(3+P)](\pi^2+\alpha^2) - a_{22}\alpha^2 A\pi[3\pi^2 + \alpha^2 + \tfrac{1}{4}P(\pi^2+\alpha^2)] + (1 - a_{13})(\alpha^2 A \pi)^2 \tfrac{1}{16}[1 - (\alpha^2+\pi^2)/4\pi^2] + c_{11}\alpha^4 \pi A \tfrac{1}{4}(1+P)(\pi^2+\alpha^2). \tag{3.16}$$

As before, terms of the order P^2 have been neglected. Inspection of (3.16) shows that for $\alpha^2 \approx \frac{1}{2}\pi^2$ the growth rate σ_2 is negative if the amplitude A is so small that all terms but the first on the right-hand side can be neglected. An expansion in powers of the amplitude A yields

$$2\sigma_2(1+P) = \frac{\pi^2}{\alpha^2} - 3 - P + \frac{(\alpha\pi A)^2}{\pi^2+\alpha^2}\left(\tfrac{5}{16} + \tfrac{1}{4}P - \frac{\alpha^2+\pi^2}{64\pi^2}\right) + (\alpha\pi A)^4 \left\{ \frac{\alpha^2[1 - (\alpha^2+\pi^2)/4\pi^2]\{\frac{1}{16}[(\alpha^2+\pi^2)/4\pi^2 - 1] + \frac{1}{10}(3\pi^2+\alpha^2)[1/(\pi^2+\alpha^2) + P/2(9\pi^2+\alpha^2)] + \frac{1}{40}P\}}{(9\pi^2+\alpha^2)^2(\pi^2+\alpha^2) - (\pi^2+\alpha^2)^4(9\pi^2+\alpha^2)^{-1}} - \frac{1+P}{32(\pi^2+\alpha^2)^2} \right\} + \dots \tag{3.17}$$

Because the convergence is not rapid enough the expansion (3.17) cannot be used to calculate the critical amplitude A_i for which σ_2 vanishes. To achieve this goal the coefficients a_{13}, a_{22} and c_{11} have been determined numerically in appendices A and B in the case $\alpha^2 = \frac{1}{2}\pi^2$, $P = 0$. The result is

$$\sigma_2 \geqslant 0 \quad \text{for} \quad A \geqslant A_i \equiv 1{\cdot}215. \tag{3.18}$$

Each of the terms on the right-hand side of (3.16) with the exception of the first gives a positive contribution to σ_2. For this reason it is not possible to trace the cause of the instability to any particular term in the equations. Expression (3.17) indicates that the Prandtl number does not influence the critical amplitude A_i very much, although this statement may have to be changed when terms of the order P^2 and higher are taken into account. The experimental observations which will be discussed in § 5 tend to support the notion that the instability depends predominantly on the parameter A. Since the rate of growth of the instability increases like b^2 for low values of b the strongest growing disturbance for a given value of $A > A_i$ will correspond to a finite value of b, even though we anticipate that the lowest value of A at which the instability occurs is given by A_i, corresponding to the limit $b \to 0$. In order to determine the wavenumber of maximum growth the contributions to σ of higher order in b have to be taken into account. Instead of pursuing the expansion analysis further we shall use the Galerkin analysis for this purpose.

4. Galerkin analysis

In this section we shall use a more general, yet less systematic, technique for the analysis of the eigenvalue problem posed by the stability equations (2.10). For simplicity we assume $P = 0$, in which case $\tilde{\theta}$ can be expressed in terms of $\tilde{v}$. Without making any assumptions about the parameter b solutions of equations (2.10) can be obtained by expanding $\tilde{v}$ and $\tilde{w}$ in terms of a system of orthogonal functions:

$$\left.\begin{aligned} \tilde{v} &= \sum_{n,m} f_{nm} v_{nm}, \\ \tilde{w} &= \sum_{n,m} g_{nm} \partial_z v_{nm} + \sum_{m} g_{0m} e^{im\alpha x}, \end{aligned}\right\} \qquad (4.1)$$

with $v_{nm} \equiv \sin n\pi z \, e^{im\alpha x}$. The summation runs over the indices $n = 1, \ldots, \infty$, $m = -\infty, \ldots, -1, 0, 1, \ldots, \infty$. After introducing the representation (4.1) into (2.10) we obtain a homogeneous system of equations for the unknown coefficients f_{nm} and g_{nm} by multiplying (2.10a) and (2.10b) by v_{nm} and $\partial_z v_{nm}$, respectively, and averaging the result over the fluid layer. The homogeneous system of equations has a solution when the determinant of the matrix of the coefficients multiplying the unknowns f_{nm} and g_{nm} vanishes. This solvability condition determines the eigenvalue σ.

Since this method of analysis is often used in stability problems and since the expressions for the matrix elements are rather lengthy, we shall not describe the solution of the problem explicitly. We note that the system of equations for f_{nm} and g_{nm} separates into two systems corresponding to either even values or odd values of $n+m$. Both systems can be separated even further using the symmetry of the stationary solution with respect to the x direction. In order to evaluate the determinant, the matrix of the coefficients of the unknowns f_{nm} and g_{nm} has to be truncated. To include only the most important modal interactions we neglect unknowns and equations corresponding to indices with $|m|+n \geqslant 4$, $|m| \geqslant 2$ or $n \geqslant 3$. In this case the determinant reduces, owing to the separation property, to three determinants of 4×4 matrices and one determinant of a 3×3 matrix. The two determinants corresponding to odd values of $m+n$ do not yield eigenvalues with positive real part in the case of interest $\alpha = \alpha_c \equiv \pi/\surd 2$. The only instability in this case corresponds to disturbances which are antisymmetric in x and have even values of $m+n$ as is to be expected for the oscillatory instability. The real and imaginary parts of the growth rate σ obtained from the determinant of the 4×4 matrix are plotted in figure 1. The imaginary part matches the value (3.11) obtained from the analytical theory almost exactly as b approaches zero. Owing to the truncation the real part of the growth rate is not as well approximated. In place of the accurate relation (3.18) the Galerkin

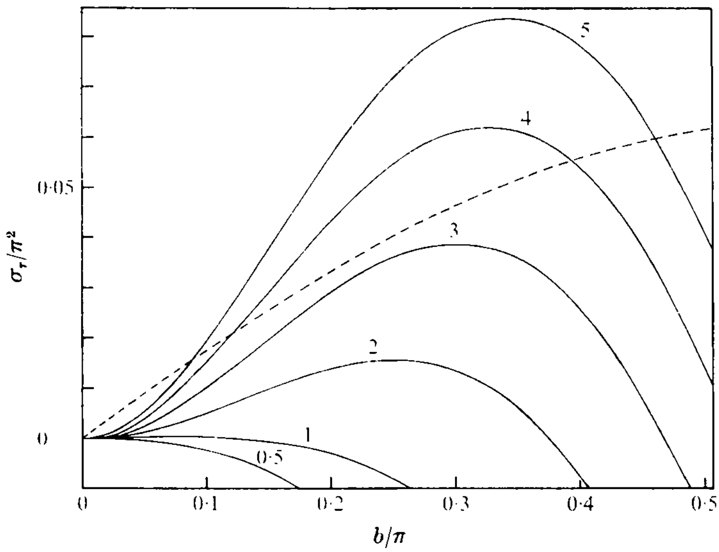

FIGURE 1. The dependence of the real and imaginary parts of the growth rate, σ_r and σ_i, on the wavenumber b and amplitude A. ——, σ_r, values of A indicated by the numbers by the curves; - - -, $\frac{1}{2}\sigma_i/A\pi^3$ for the case $\sigma_r = 0$.

procedure yields $\sigma_r \geqslant 0$ for $A \geqslant 0{\cdot}967$ in the limit $b \to 0$. Since the critical amplitude of the instability has been obtained already in (3.18) and since at this point we are more interested in the qualitative dependence of σ_r on b, no attempt has been made to replace the analytical evaluation of the determinant by a more sophisticated computation at high truncation values. Figure 1 shows that the wavenumber b_m of the disturbance with maximum real part σ_r increases strongly from zero as the amplitude A increases from the critical value. As the amplitude increases further the value b_m approaches a limiting value in the neighbourhood of $0{\cdot}38\pi$. This behaviour is in agreement with the experimental observation that the instability always occurs at a finite value of b.

5. Discussion

From the phenomenological point of view the oscillatory instability of rolls resembles a wave propagating along a rope. The roll pattern is shifted perpendicular to its axis periodically along the axis and in time. The oscillatory behaviour of the translation is caused by the z-independent component of the vertical vorticity which in turn is generated by the interaction between the disturbance and the basic flow. Since the frequency of oscillation may have both signs, according to (3.10), the instability can manifest itself as a wave travelling in either direction along the axis of the rolls or as a standing wave. A qualitative picture of the oscillating roll pattern is shown in figure 2.

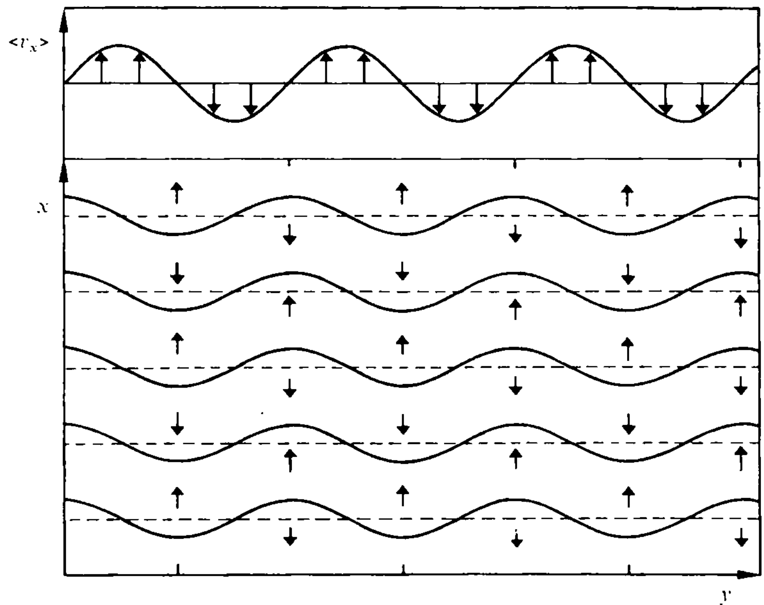

FIGURE 2. Qualitative picture of oscillating convection rolls corresponding to a wave travelling in the positive y direction. The x, z average of the horizontal velocity shown at the top of the figure (described by $\langle w_1 \rangle$ in the analysis) has the opposite sign in the case of a wave travelling in the negative y direction.

The phenomenological picture of the instability compares well with the detailed observations by Willis & Deardorff (1970) of the oscillations of convection rolls in a layer of air. These authors stress the point that the oscillations occur essentially independently of depth, in contradiction to earlier theoretical explanations. Because of its smaller Prandtl number, convection in mercury would probably offer a better opportunity for quantitative comparison with the theory. Unfortunately, however, no detailed observations of these oscillations in mercury have been reported in the literature. In the following we shall attempt to apply the theory to the experimental situation, even though the limit of small Prandtl number and the free boundary conditions are not approached in any of the experiments. Since the experimental results are given in terms of the Rayleigh number, we use the last relation in (2.7) to express the amplitude as $R-R_c$ with $R_c = (\pi^2+\alpha_c^2)^3/\alpha_c^2$. Accordingly, the condition (3.18) for instability assumes the form

$$\frac{R-R_c}{R_c} \geqslant \frac{R_i-R_c}{R_c} \equiv 0{\cdot}310P^2. \tag{5.1}$$

The period τ of the oscillations measured in units of d^2/κ as in the experiments is given by

$$\tau = P^{-1}\frac{2\pi}{b\,|\sigma_1|} = \frac{2}{3^{\frac{1}{2}}b}\left(\frac{R-R_c}{R_c}\right)^{-\frac{1}{2}}. \tag{5.2}$$

It is not surprising that the observations of Willis & Deardorff (1970) and Krishnamurti (1970) indicate much higher Rayleigh numbers for the occurrence of instability than those suggested by (5.1). For a rigid boundary exerts a stabilizing influence on the mechanism of instability and, in addition, the amplitude of the velocity field increases less rapidly with the Rayleigh number in the case of rigid boundaries. The experimental value of the critical Rayleigh number for the onset of oscillations has not yet been well-determined as a function of the Prandtl number. The observations show, however, that the critical value increases strongly with the Prandtl number, at least for $P < 10$.

A much closer correspondence exists between the theoretical expression (5.2) and the observed values of the period of oscillation. The dependence on the Rayleigh number lies between the observed power laws of $R^{-0 \cdot 4}$ (Willis & Deardorff 1970) and $R^{-0 \cdot 64}$ (Rossby 1969; Krishnamurti 1970). The fact that the wavenumber b observed by Willis & Deardorff is larger than the wavenumber α of the basic rolls appears to be in disagreement with the result of § 4. However, owing to the high Rayleigh number the wavenumber of the rolls in this experiment is only $0 \cdot 57\alpha_c$ if it is expressed in terms of the critical wavenumber for rigid boundaries. To investigate the dependence of the wavenumber of strongest growth, b_m, on α the calculations described in § 4 have been carried out for a number of values α different from α_c. The results show that the wavenumber b_m of maximum growth divided by its value given in figure 1 increases at roughly half the rate at which α/α_c decreases from the value 1. Hence the observed wavenumber $b = 0 \cdot 8\alpha_c$ is in near agreement with the prediction of the theory if we assume that the results for rigid and for free boundaries become roughly the same if expressed in terms of the respective critical wavenumber α_c. The fact that the period of oscillation is independent of the Prandtl number according to (5.2) is demonstrated by a graph shown in Krishnamurti's paper in which measurements ranging over several orders of magnitude of the Prandtl number have been plotted.

At Prandtl numbers of order 10 and larger, the oscillations occur as the instability of a three-dimensional stationary convection pattern since the Rayleigh number exceeds 23 000. This value marks the upper limit beyond which two-dimensional convection in the form of rolls is unstable and is superseded by the three-dimensional structure of bimodal convection (Busse & Whitehead 1971). The qualitative agreement between observations at widely different Prandtl numbers indicates that the mechanism leading to the oscillatory instability operates in the case of three-dimensional convection in a way similar to that in the case of rolls. Although this interpretation needs further investigation, it eliminates the need for the explanation of the oscillations in terms of Howard's theory of the instationary thermal boundary layer (Rossby 1969) or in terms of Welander's thermal oscillations (Krishnamurti 1970). Shortcomings of both explanations have been pointed out in the detailed investigation of Willis & Deardorff. These authors refer also to numerical calculations by Lipps which indicate oscillations of rolls. Unfortunately, no written account of this study seems to be available.

6. Concluding remarks

An important result of the preceding analysis is the fact that the oscillatory instability of rolls is caused solely by the action of the hydrodynamic advection terms in the equations of motion. The mechanism of instability is thus independent of the release of gravitational energy which produces the convection rolls. We expect for this reason that any field of two-dimensional vortices in the form of rolls can become unstable by the same mechanism of instability.

The oscillatory instability of convection rolls is related, at least phenomenologically, to the non-axisymmetric instability of Taylor vortices between concentric cylinders rotating at different speeds. However Taylor vortices differ from convection rolls by the additional azimuthal component of the velocity field. In their theoretical analysis of the problem, Davey, DiPrima & Stuart (1968) tend to support the view originally expressed by Meyer (1966) that the instability of the Taylor vortices is caused by an Orr–Sommerfield type instability of the azimuthal velocity component. Because of the anistropy between the azimuthal and axial direction and because of the additional azimuthal velocity component, it is rather difficult to isolate the mechanism for the Taylor vortex instability. Hence the question of its relation to the mechanism of oscillatory instability cannot be answered definitely at this point.

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Appendix A

The adjoint homogeneous problem for the system of equations (3.11 a,b,d) together with the boundary conditions (2.6) can be written in the form

$$\nabla^4\Delta_2 v^* - \Delta_2\theta^* + V_a^+ v^* - iV_b^+\overline{w}^* + R^{-1}P\boldsymbol{\delta}.(\theta\nabla\theta^*) = 0, \qquad (\text{A }1a)$$

$$\partial^2_{zz}\overline{w}^* + iV_c^+ v^* = 0, \qquad (\text{A }1b)$$

$$\nabla^2\theta^* - R\Delta_2 v^* + P\boldsymbol{\delta}v.\nabla\theta^* = 0, \qquad (\text{A }1c)$$

where the operators V_a^+, V_b^+ and V_c^+ are defined by

$$V_a^+ v^* \equiv \boldsymbol{\delta}.(\boldsymbol{\delta}v.\nabla\boldsymbol{\delta}v^*) + \boldsymbol{\delta}.(\boldsymbol{\delta}v.\boldsymbol{\delta}\nabla v^*),$$

$$V_b^+ \overline{w}^* \equiv \partial_z(\partial_z\overline{w}^*\partial_x\Delta_2 v) + \partial_z\overline{w}^*\partial^2_{zx}\Delta_2 v,$$

$$V_c^+ v^* \equiv \overline{v^*\nabla^2\Delta_2\partial_x v - \langle v^*\nabla^2\Delta_2\partial_x v\rangle - \partial^2_{zz}(v^*\Delta_2\partial_x v)}.$$

The condition that the z average vanishes has been imposed in determining $V_c^+ v^*$ since the same requirements hold for $\overline{w}_n'$. The variables v^*, θ^* and $\overline{w}^*$ have to satisfy the same boundary conditions (2.6) as v, θ and w, respectively. To solve equations (A 1) we assume the representation

$$\left.\begin{aligned} v^* &= \sum_{m,n} a_{mn}\sin m\alpha x\sin n\pi z, \\ \theta^* &= \sum_{m,n} b_{mn}\sin m\alpha x\sin n\pi z. \end{aligned}\right\} \qquad (\text{A }2)$$

Integration of (A 1*b*) yields the expression

$$\overline{w}^* = \tfrac{1}{4} i\alpha^3 \sum_{n\geqslant 1} \cos n\pi z (a_{1n+1} - a_{1n-1}) \, [1 - (\alpha^2 + \pi^2)/n^2\pi^2]. \qquad \text{(A 3)}$$

Since the solution v_0, θ_0 of the adjoint homogeneous system is antisymmetric in x we have assumed the same property in the representation (A 2) for v^*, θ^*. The symmetric solution of (A 1) will not yield any additional constraint in the solvability condition (3.12).

Insertion of the expressions (A 2) and (A 3) into (A 1*a*, *d*), multiplication of the equations by $\sin \mu\alpha x \sin \nu\pi z$ and subsequent averaging yields a system of linear equations for the unknown coefficients $a_{\mu\nu}$ and $b_{\mu\nu}$:

$$-\mu^2(\nu^2\pi^2 + \mu^2\alpha^2)^2 a_{\mu\nu} + \mu^2 b_{\mu\nu} + \frac{A\pi}{4}\alpha^2 \sum_{S_1, S_2} a_{\mu+S_1 \nu+S_2} \mu(\mu + S_1)$$
$$\times [\pi^2(\nu^2 - 1) + \alpha^2(\mu^2 - 1)(S_1\mu - S_2\nu)] - \frac{A^2\alpha^4\pi^2}{8}(\nu^2 - 1)\left\{(a_{1\nu} - a_{1\nu-2})\left(1 - \frac{\alpha^2+\pi^2}{(\nu-1)^2\pi^2}\right)\right.$$
$$\left. + (a_{1\nu} - a_{1\nu+2})\left(1 - \frac{\alpha^2+\pi^2}{(\nu+1)^2\pi^2}\right)\right\}\delta_{1\mu} + \frac{PA}{R}\frac{\pi\mu(a^2+\pi^2)^2}{4} \sum_{S_1, S_2} b_{\mu+S_1\nu+S_2}(S_1 S_2 \nu - \mu) = 0, \qquad \text{(A 4a)}$$

$$-(\nu^2\pi^2 + \mu^2\alpha^2) b_{\mu\nu} + R(\mu\alpha)^2 a_{\mu\nu} + P\frac{A\alpha^2\pi}{4} \sum_{S_1, S_2} b_{\mu+S_1\nu+S_2}(S_1\mu - S_2\nu) = 0. \qquad \text{(A 4b)}$$

The symbols S_1, S_2 take either the value ± 1; the sum has to be extended over those four possibilities. The solutions of (A 4) separate into solutions for which $\mu + \nu$ is an odd integer and for which $\mu + \nu$ is an even integer. We shall restrict the attention to the latter possibility since it corresponds to the symmetry property of the inhomogeneity in (3.11). Equations (A 4) for $\mu = \nu = 1$ are not coupled to the rest of the equations and can be solved readily by making the following choice of the undetermined amplitude:

$$a_{11} = 1, \quad b_{11} = (\pi^2 + \alpha^2)^2. \qquad \text{(A 5)}$$

After these expressions have been inserted, (A 4) for $\nu + \mu > 2$ forms an infinite system of inhomogeneous equations for the unknowns $a_{\mu\nu}$ and $b_{\mu\nu}(\nu + \mu > 2)$. Of particular interest are the coefficients a_{13}, b_{13}, a_{22} and b_{22}, which are needed for the evaluation of the solvability condition (3.12). To obtain a solution of the problem, the system (A 4) with $\nu + \mu > 2$ can be solved by using an expansion in powers of the amplitude A or by truncating the system. The first method yields

$$a_{13} = (A\pi)^2 \frac{\alpha^4[1 - (\alpha^2 + \pi^2)/4\pi^2]}{(9\pi^2 + \alpha^2)^2 - (\pi^2 + \alpha^2)^3 (9\pi^2 + \alpha^2)^{-1}} + \ldots,$$

$$b_{13} = a_{13}\frac{(\pi^2 + \alpha^2)^3}{\alpha^2 + 9\pi^2} + \ldots,$$

$$a_{22} = \frac{-a_{13}\alpha^2}{10(\pi^2 + \alpha^2)} A\pi \left(1 + \frac{P}{2}\frac{\pi^2 + \alpha^2}{9\pi^2 + \alpha^2}\right) + \ldots,$$

$$b_{22} = (\pi^2 + \alpha^2) a_{22} - PA \frac{b_{13}\alpha^2}{4(\pi^2 + \alpha^2)} + \ldots.$$

The most natural way of truncation seems to be the neglect of all unknowns and equations where the sum $\mu+\nu$ of the indices is larger than some prescribed value N. The coefficients plotted in figure 3 have been computed by this method in the

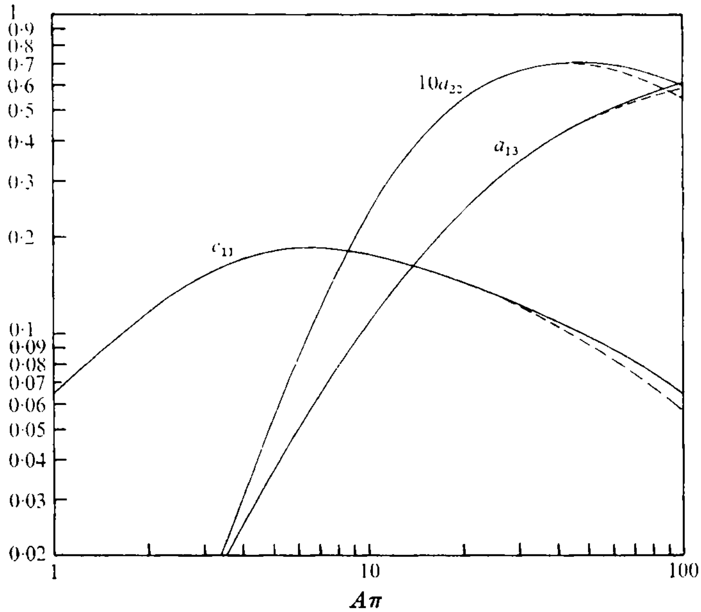

FIGURE 3. The coefficients a_{13}, a_{22} of the solution (A 2) of the adjoint homogeneous problem in the case $\alpha^2 = \frac{1}{2}\pi^2$, $P = 0$, and the coefficient c_{11} of the solution (B 1) for w_2'. Coefficients $a_{\mu\nu}$, $c_{\mu\nu}$ with $\mu+\nu \geqslant N = 12$ have been neglected in the computations. The dashed lines indicate the deviations of the results when the truncation number $N = 10$ is used.

case $P = 0$, $\alpha^2 = \frac{1}{2}\pi^2$. The convergence of the solutions for different values of N is very good for the moderate values of A which are relevant to the question of instability.

Appendix B

In order to solve (3.11c) a Fourier representation for w_2' of the form

$$w_2' = \langle w_1 \rangle \sum_{\nu\geqslant 0,\, \mu\geqslant 1} c_{\mu\nu} \cos \mu\alpha x \cos \nu\pi z \qquad \text{(B 1)}$$

will be assumed. Each Fourier component satisfies the boundary condition (2.6). The symmetry of (3.11c) shows that only Fourier components which are symmetric in x enter the representation and that the coefficients $c_{\mu\nu}$ are vanishing unless the sum of the indices, $\mu+\nu$, is an even integer. Using the representation (B 1) equation (3.11c) can be reduced to the following system of linear algebraic equations:

$$c_{\mu\nu}(\nu^2\pi^2+\mu^2\alpha^2)\mu^2 + \tfrac{1}{4}\mu A\alpha^2\pi \sum_{S_1 S_2} c_{\mu+S_1\nu+S_2}(\mu+S_1)(S_1\mu - S_2\nu) = \pi\delta_{\mu 1}\delta_{\nu 1}, \qquad \text{(B 2)}$$

where the sum has to be extended over the four possibilities of the signs $S_1 = \pm 1$, $S_2 = \pm 1$. For the determination of σ_2 only the coefficient c_{11} has to be known. An expansion in terms of powers of the amplitude A yields as solution of (B 2)

$$c_{11} = \pi A(\pi^2+\alpha^2)^{-1}\left(1-\frac{1}{8}\frac{\alpha^2}{\pi^2+\alpha^2}(\pi A)^2+\ldots\right). \tag{B 3}$$

In analogy with the solution of the adjoint problem in the appendix A we obtain an alternative solution in the case $\alpha^2 = \frac{1}{2}\pi^2$ by truncating the system (B 2) for $\mu+\nu \geqslant N$. The dependence of numerical value of c_{11} on the amplitude A is shown in figure 3 for different values of N.

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